

Modeling Turbulent Flow with Implicit LES

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Abstract

Implicit large eddy simulation (*ILES*) is a methodology for modeling high Reynolds' number flows that combines computational efficiency and ease of implementation with predictive calculations and flexible application. Although *ILES* has been used for more than fifteen years, it is only recently that significant effort has gone into providing a physical rationale that speaks to its capabilities and its limitations. In this talk, we will present new theoretical results aimed toward building a justification and discuss some remaining gaps in our understanding and our practical application of the *ILES* technique.

1 Introduction

Implicit Large Eddy Simulation (*ILES*) is an alternate approach to simulating high Reynolds' number flows that combines simplicity of implementation with computational efficiency, and that has been successfully applied to complex flows in engineering, geophysics and astrophysics. The underlying strategy of *ILES* is to apply fluid solvers based on nonoscillatory finite volume (NFV) approximations directly to turbulent flows; the numerical algorithms provide effective modeling of the unresolved dynamics and no explicit subgrid scale turbulence models are needed or used. The absence of any explicit parameters holds the promise of increased predictiveness in simulations. Further, the *ILES* approach offers a constructive approach to modeling more general systems of equations and coupled physics processes.

Despite these attractive advantages, *ILES* has not yet been widely accepted in the turbulence modeling community. Given its many documented successes, we surmise that it is the lack of rigorous justification that concerns the conventional modeling community. Over the past five years, we have made considerable progress in studying the *ILES* process and developing a rationale that combines the perspectives of physics and numerical analysis. In this paper, we will summarize these ideas. Our thesis can be succinctly stated as follows:

The success of ILES follows from the fact that NFV methods accurately solve the equations that describe the dynamics of finite volumes of fluid.

That finite volumes of fluid obey different equations than the continuum Navier-Stokes equations (NSE) is widely accepted in the turbulence community; the use of filtered equations as a basis for conventional LES and the appearance of explicit subgrid scale models is a clear example of this recognition. However, it is not widely known that it is possible to derive equations for the evolution of finite volumes of fluid. We will exhibit and discuss the form of the finite scale equations for 2D NSE – i.e., the equations that govern the volume average of velocity whose point values satisfy the two-dimensional Navier-Stokes equations. The equations themselves have the form of an infinite convergent series, and are a generalization of the finite-scale 1D Burgers' equations derived in [10]. One new feature that

appears in multiple dimensions is that the finite-scale equations depend on the shape of the averaging volume as well as its size.

To demonstrate that a numerical algorithm is solving the finite-scale equations, we can compare its modified equation (ME), a PDE whose solution accurately mimics the discrete solution of the algorithm [6]. The ME is derived from a Taylor expansion of the discrete equations and, like the finite-scale equations, have the form of an infinite series. We will exhibit the ME for a particular NFV algorithm, MPDATA [18] and show the qualitative agreement between its ME and the physical finite-scale equation.

We should emphasize that we do not imply that the two corresponding series are identical term by term, but only that the lowest-order truncation terms of the ME contain essential similarities to the finite-scale equations. These essential elements will be described in sections 5 and 6. From the point of view of accuracy, this is sufficient. However, for the discrete equations, the additional consideration of computational stability must be addressed. In fact, differences in the higher-order terms of the two series can have consequences. The physical finite-scale equations are globally stable, whereas the NFV equations are locally stable, a more stringent condition. This difference may affect the simulated variability of turbulent flows. We note that this is not an issue specific to *ILES*, but arises in explicit large eddy simulation as well.

We conclude this introduction with a summary of the remainder of the paper. In section 2, we will give a heuristic introduction to the finite-scale equations. In section 3, we present a brief numerical perspective, intended to show that *ILES* can be viewed as an evolutionary result of much previous and mainline research in algorithmic development for computational fluid dynamics. In particular, we show that the two crucial algorithmic elements are nonoscillatory methods and finite-volume approximation. In section 4, we explore the connection between these algorithmic elements and important realizability properties of physical systems.

In sections 5 – 7 we make our arguments more quantitative. In section 5 we present the finite scale equation for the 2D NSE and discuss features that distinguish it from the underlying continuum PDE. In section 6 we present the ME of the MPDATA algorithm, and compare it to the finite-scale equation. In section 7 we focus on the inviscid dissipation

of the finite-scale and ME equations. We conclude the paper in section 8 with a summary and a short discussion of future directions.

2 Heuristic Discussion

One criticism that is sometimes made of *ILES* is that "nature does not have a Δx ." Here, Δx is "shorthand" for the detailed information about the size and shape of a computational cell in a numerical simulation of turbulence. The implication is that since the modeling of the unresolved scales depends on the truncation error of the algorithm which is numerical and not physical, the modeling itself cannot be general. One answer to this criticism is that standard subgrid scale models also depend on Δx . However, a more satisfactory answer might begin by noting that we do not model nature, rather we attempt to predict and analyze experiments. Experiments are performed with instruments that do have intrinsic scales of length and time.

Consider measuring the velocity distributions in a turbulent fluid by means of a hot wire anemometer. For the scales of motion of the fluid are comparable or smaller than the wire diameter, the results will depend on that size. Of course, we expect that the larger scales of motion will be relatively insensitive to the wire size. In the case of fluid simulations, we are solving for the velocity (and other variables) averaged over the volume of a cell. Because of the nonlinearity of the advective terms, we should expect that the governing equations will differ from the PDEs that describe nature, and that those differences will depend on the size and shape of the volume over which we average.

The previous paragraphs are meant to justify that truncation terms *may* act as subgrid scale models, but do not address the question of whether they will be effective. At present, it is an empirical result that most NFV algorithms are effective in *ILES* simulations, but that no other class of algorithms has this property. This leads to the question of whether the success of NFV methods is happenstance, or whether there is some deeper reason. In fact, one of the earliest published articles on implicit turbulence modeling [13] answers this question by asserting "a convenient conspiracy" of physics and numerics. In the next two

sections, we will explore the ideas that underlie this conspiracy, while in sections 5 and 6, we will give those ideas quantitative substance.

3 A Numerical Perspective

The idea that the analytic PDEs may not be a sufficient model for simulating high Reynolds' flows goes back to the earliest days of computing. It was found that one-dimensional Lagrangian simulations of shocks were contaminated by unphysical oscillations behind the shock. Von Neumann and Richtmyer [21] recognized the problem that the shock was not resolved on the mesh, with the result that not enough kinetic energy was being dissipated; they proposed to add an "artificial" viscosity to the equations sufficient to smear the shock over several computational cells. A few years later, Smagorinsky generalized this idea to multidimensional simulations of weather in an Eulerian framework [16]. An interesting account of the relationships of these two works can be found in [17].

From these works, and the huge literature on shock capturing and subgrid scale modeling that followed, two important conclusions have emerged.

- The form of the artificial viscosity, or the subgrid scale model, is nonlinear. That is, the coefficient of viscosity depends on the solution. From a scaling point of view, the artificial viscosity depends on Δx^2 .
- There is a critical magnitude of viscosity required to eliminate unphysical oscillations. In modern parlance, this is termed an entropy condition.

The importance of nonlinear approximations was also highlighted in the famous theorem of Godunov (see [8]). However, it would take another fifteen years for the CFD community to understand the practical import of nonlinearity.

In a 1969 paper [7], Hirt pointed out that subgrid scale models used in turbulence simulations have the same order of magnitude as the truncation errors of a second-order accurate fluid solver – both scale as Δx^2 . The aim of that paper was to emphasize the importance of accurate fluid solvers. However, one might turn the conclusion around and ask whether it is possible to design an algorithm whose truncation errors could function as the subgrid scale model. Unfortunately, in 1969 NFV algorithms had not yet been invented.

In the early 1980s, Belotserkovskii [1] proposed his "principle of stable solution," which comes very close to *ILES*. Belotserkovskii emphasized the importance of flux algorithms (viz finite volume) and of "monotonizing" algorithms. He proposed mixing a low-order donor cell scheme with a high-order centered difference. In fact, many modern NFV schemes are constructed in this way using a nonlinear (i.e., flow dependent) mixing coefficient. However, Belotserkovskii used a constant mixing coefficient, ending with what is sometimes termed interpolated donor cell, which is only first-order accurate.

The early 1970s saw the introduction of the first NFV schemes, the Flux Corrected Transport (FCT) method of Boris and Book [2] and the MUSCL schemes of van Leer [20], and Collela and Woodward [5]. It is no accident that some of these early pioneers of monotonicity-preserving methods became the pioneers of *ILES*. Jay Boris is generally credited with the earliest recognition and use of FCT for implicit turbulence modeling, a technique he termed MILES for monotonically integrated large eddy simulation [3]. An expanded recounting of the early history of *ILES* can be found in [10], which illustrates the diversity both of successful NFV algorithms and of validated areas of application.

To summarize this brief numerical history, both the difficulties and the successful strategies of modeling high Reynolds' flows are inherent in the nonlinearity of the Navier-Stokes equations, and in particular of the advective terms. It is found empirically that both the nonoscillatory and the finite-volume attributes are necessary for effective *ILES*. Next, we will explore the connection of these numerical ideas to physics.

4 A Physical Perspective

The connection between the mathematical constraint of preserving monotonicity and physics was made in an interesting, but little known paper [12]. Merriam showed that the preservation of monotonicity implies compliance with the second law of thermodynamics within a computational cell. However, a computational cell is not a closed system, and so monotonicity preservation is a sufficient, but not a necessary condition to ensure the nondecrease of entropy in the physical system. This was pointed out in [10], where it was suggested

that relaxing the monotonicity constraint might produce simulations with more realistic variability.

Finite-volume approximations of the advective terms, also known as flux form, ensure exact conservation of the advected variables *by construction*. That is, finite-volume algorithms are conservative to the level of roundoff error whereas the alternative advective form approximations are conservative to only to the level of the truncation error. This gains significance when one recalls Hirt’s observation [7] that these truncation errors are of the same order of magnitude as the turbulent subgrid scale terms that model the effects of the unresolved scales on the resolved scales. A second feature of finite-volume approximations is that they assure that the fluxes can be written as the divergence of a tensor. In turn, this guarantees that the truncation terms, and hence the implicit subgrid scale tensor, also can be written as the divergence of a tensor. We will verify this property explicitly in section 6 for a particular NFV scheme.

As a matter of interpretation, finite-volume schemes solve for the volume-averaged variables whereas advective schemes solve for the point quantities. It may be assumed that the volume-averaged quantities vary more smoothly in space than the underlying point distributions, and hence are more amenable to discrete approximation. However, whether they are smooth enough to ensure convergence of the finite-scale equations remains an open theoretical question. From a more practical point of view, the existence of convergent governing equations is a prerequisite to computing turbulent flows and substantial evidence exists to confirm the effectiveness of numerical simulation.

5 Finite-Scale Equations

The Navier-Stokes equations for the two-component velocity vector $U = (u, v)$ are

$$\begin{aligned} \frac{\partial u}{\partial t} &= -(uu)_x - (uv)_y - P_x + \nu(u_{xx} + u_{yy}) \\ \frac{\partial v}{\partial t} &= -(uv)_x - (vv)_y - P_y + \nu(v_{xx} + v_{yy}), \end{aligned} \tag{5.1}$$

plus the equation of incompressibility

$$u_x + v_y = 0. \tag{5.2}$$

Here, P is the pressure and ν is the coefficient of viscosity. We note that the pressure is a diagnostic variable, and can be found by solving an elliptic equation that enforces incompressibility.

We will define the volume-averaged velocities

$$\bar{u}(x, y) \equiv \frac{1}{\Delta x \Delta y} \int_{x-\frac{1}{2}\Delta x}^{x+\frac{1}{2}\Delta x} \int_{y-\frac{1}{2}\Delta y}^{y+\frac{1}{2}\Delta y} u(x', y') dx' dy' \quad (5.3)$$

and

$$\bar{v}(x, y) \equiv \frac{1}{\Delta x \Delta y} \int_{x-\frac{1}{2}\Delta x}^{x+\frac{1}{2}\Delta x} \int_{y-\frac{1}{2}\Delta y}^{y+\frac{1}{2}\Delta y} v(x', y') dx' dy' \quad (5.4)$$

That is, here we have chosen a specific volume of integration, a rectangle, that mimics a computational cell in a regular mesh.

We note that (5.2) is linear. Hence the spatial differentiation and the volume averaging commute, and it immediately follows that

$$\bar{u}_x + \bar{v}_y = 0. \quad (5.5)$$

Similar arguments apply to the time derivatives and the viscous terms in (5.1). However, the nonlinearity of the advective terms requires more care. Our results below use the same renormalization arguments described in detail in [10], but extended to two spatial dimensions. We note that a critical assumption in the renormalization process concerns the smoothness of the averaged variables.

The final result for the finite-scale (volume-averaged) momentum equations is

$$\begin{aligned} \frac{\partial \bar{u}}{\partial t} = & - (\bar{u}^2)_x - (\bar{v}\bar{u})_y - \bar{P}_x + \nu (\bar{u}_{xx} + \bar{u}_{yy}) \\ & - \frac{1}{3} \left(\frac{\Delta x}{2} \right)^2 [(\bar{u}_x \bar{u}_x)_x + (\bar{v}_x \bar{u}_x)_y] - \frac{1}{3} \left(\frac{\Delta y}{2} \right)^2 [(\bar{u}_y \bar{u}_y)_x + (\bar{u}_y \bar{v}_y)_y] + HOT \end{aligned} \quad (5.6)$$

$$\begin{aligned} \frac{\partial \bar{v}}{\partial t} = & - (\bar{u}\bar{v})_x - (\bar{v}^2)_y - \bar{P}_y + \nu (\bar{v}_{xx} + \bar{v}_{yy}) \\ & - \frac{1}{3} \left(\frac{\Delta x}{2} \right)^2 [(\bar{u}_y \bar{u}_y)_x + (\bar{u}_y \bar{v}_y)_y] - \frac{1}{3} \left(\frac{\Delta y}{2} \right)^2 [(\bar{u}_y \bar{v}_y)_x + (\bar{v}_y \bar{v}_y)_y] + HOT \end{aligned} \quad (5.7)$$

Here, HOT denotes additional terms of higher order in Δx and Δy . The equation for the pressure \bar{P} is found by applying the incompressibility constraint (5.5) to the momentum equations. We note that as $\Delta x, \Delta y \rightarrow 0$, the finite-scale equations limit properly to the 2D

NSE. However, in simulating high Reynolds' number flows, Δx and Δy *never* go to zero. Also, we emphasize that (5.6) and (5.7) *are specific* to the rectangular volume of integration.

Let us now make several observations about the finite-scale equations (5.6) and (5.7). First, we mention that the derivation is a straightforward extension of that in [10], and that there is no obstacle to further extending the results to three spatial dimensions and to include time averaging as well.

Second, we note that the lowest-order finite-scale corrections are quadratic in the mesh spacings Δx and Δy and that there are no terms of order $\Delta x \Delta y$. The result does depend on both the size and shape of the volume chosen for averaging. We have chosen rectangles in two dimensions to emphasize the comparisons with our numerical results in the next section.

Third, we note that the finite-scale corrections have form of the divergence of a symmetric tensor, which implies the conservation of the finite-scale momentum. In the language of LES, the finite-scale corrections correspond to the divergence of a subgrid scale tensor $\nabla \cdot \tau$ where

$$\tau^{xx} = -\frac{1}{3} \left[\left(\frac{\Delta x}{2} \right)^2 \bar{u}_x^2 + \left(\frac{\Delta y}{2} \right)^2 \bar{u}_y^2 \right] + HOT \quad (5.8)$$

$$\tau^{xy} = \tau^{yx} = -\frac{1}{3} \left[\left(\frac{\Delta x}{2} \right)^2 \bar{u}_x \bar{v}_x + \left(\frac{\Delta y}{2} \right)^2 \bar{u}_y \bar{v}_y \right] + HOT \quad (5.9)$$

$$\tau^{yy} = -\frac{1}{3} \left[\left(\frac{\Delta x}{2} \right)^2 \bar{v}_x^2 + \left(\frac{\Delta y}{2} \right)^2 \bar{v}_y^2 \right] + HOT \quad (5.10)$$

Here the (cartesian) tensor indices are shown as superscripts to distinguish them from the subscripts that denote spatial differentiation.

Fourth, we emphasize the similarity of this subgrid scale tensor to the LES model of Clark – see, e.g., page 628 in [14]. The Clark model belongs to the class of similarity models, which are found to accurately represent the nonlinear dynamics and energy transfer of turbulence in simulations, but which are generally not sufficiently dissipative. The Clark term is often used in conjunction with a Smagorinsky term to form mixed models [14].

Finally, we call attention to the fact that the finite-scale equations depend sensitively on the length scales of the averaging. We would emphasize that these are not intrinsic scales of the flow, but in fact *represent the length scales of the observer*. The fact that

the equations and their solutions change as the length scales change is not a flaw, but a necessary consequence of their interpretation as a model of reality, as measured by a particular observer.

6 MPDATA Modified Equation

In this section, we compare the finite-scale equation derived in the previous section with the modified equation of a particular NFV scheme. MPDATA has been used successfully in *ILES* simulations of the atmosphere, both on mesoscale [9] and global scale [19] problems. It has also been successfully employed in more generic simulations of decaying turbulence in a periodic box [11]. MPDATA [18] is constructed directly using the properties of iterated upwinding, in contrast to the majority of NFV schemes which are based on the idea of flux limiting. Nevertheless, MPDATA's properties, as exposed by modified equation analysis, are typical of many NFV schemes.

Modified equation analysis (MEA) is a technique for generating a PDE whose solution closely approximates the solution of a numerical algorithm. MEA was first introduced by Hirt [6] as a heuristic means to study the stability of a numerical algorithm. MEA is based on Taylor series expansion, and so is capable of studying the properties of nonlinear equations, in contrast to von Neumann stability analysis which is based on Fourier analysis and so restricted to linear or linearized algorithms. This is an important advantage, since the nonoscillatory properties of NFV schemes are a unique property of nonlinear approximation – cf. Godunov's theorem, see p. 174 in [8].

We will assume a simple data structure where both components of velocity are located at the cell centers. A straightforward, if somewhat tedious, analysis of the MPDATA algorithm applied to the 2D Navier-Stokes equations (5.1) leads to the modified equation. Here, we focus only on the semi-discrete equations by letting the time step $\Delta t \rightarrow 0$, to correspond to the finite scale equations (5.6) and (5.7) where we did only spatial averaging. Also, we exhibit only the truncation terms originating in the advective terms, to allow direct comparison with the finite-scale "subgrid stress" terms in (5.8) – (5.10). It turns out that these truncation terms also can be written as the divergence of a tensor. To lowest order,

the implicit subgrid stress \mathcal{T} is:

$$\mathcal{T}_{xx} = \left(\frac{1}{4} u_x |u_x| + \frac{1}{12} u_x u_x + \frac{1}{3} u u_{xx} \right)_x \Delta x^2 \quad (6.1)$$

$$\mathcal{T}_{xy} = \left(\frac{1}{4} u_y |v_y| + \frac{1}{12} u_y v_y + \frac{1}{6} (u v_{yy} + v u_{yy}) \right)_y \Delta y^2 \quad (6.2)$$

$$\mathcal{T}_{yx} = \left(\frac{1}{4} v_x |u_x| + \frac{1}{12} u_x v_x + \frac{1}{6} (u v_{xx} + v u_{xx}) \right)_x \Delta x^2 \quad (6.3)$$

and

$$\mathcal{T}_{yy} = \left(\frac{1}{4} v_y |v_y| + \frac{1}{12} v_y v_y + \frac{1}{3} v v_{yy} \right)_y \Delta y^2 \quad (6.4)$$

Let us now do a detailed comparison between the finite-scale subgrid scale stress of equations (5.8) – (5.10) and the implicit subgrid scale stress of equations (6.1) – (6.4).

First of all, we note that the truncation terms can be written as a second-order tensor. This is a direct consequence of the finite-volume nature of the approximation and illustrates the importance of the "FV" in NFV methods.

Second, we note that each of the components in \mathcal{T} is quadratic in Δx or Δy , similar to the properties of τ . This is a direct consequence of the second-order accuracy of *MPDATA* and explains why first-order schemes such as donor cell are not suitable for *ILES*, even though they are nonoscillatory. Perhaps of equal importance is the implication that higher-order (than second) schemes will not have the proper dimensional dependence and also will be unsuitable for *ILES*.

Third, we note that \mathcal{T} is not symmetric in its off-diagonal components, whereas τ is symmetric. More generally, we may note the lack of certain terms in \mathcal{T} that are present in τ . For example, there are no terms of order Δy^2 in \mathcal{T}_{xx} . This source of this deficit is easy to uncover, and in fact results from the particular form used by *MPDATA* to estimate the velocity at the center of the edge of a computation cell; specifically, the average of the two values of the adjacent cells. This ignores the perpendicular variation of these values. This is also a relatively easy deficiency to fix. In computational experiments, however, we have seen little difference when "fuller" stencils are used for this averaging, indicating the relative lack of importance of these terms.

Fourth, we note the common appearance of certain terms in both subgrid scale tensors. For example, the term containing $(u_x u_x)$ in both \mathcal{T}_{xx} and τ_{xx} , and the term containing

$(u_y u_y)$ in both \mathcal{T}_{xy} and τ_{xy} . These are the critical terms from the point of view of nonlinear dynamics. We note that these terms are also similar to those appearing in self-similar explicit subgrid scale models [14].

Finally, we note the presence of certain terms in \mathcal{T} not present in τ . Some of these, e.g., the term $(u u_{xx})$ in \mathcal{T}_{xx} can be seen to be equivalent to $(u_x u_x)$ in terms of their effect on dissipation. Other new terms, e.g., $(u_x |u_x|)$, play a critical role in the area of computational stability.

7 Energy Analysis

The comparison of the finite-scale equations and modified equations of MPDATA can be extended by looking at energy dissipation. The total rate of inviscid energy dissipation – i.e., independent of the physical viscosity ν – is

$$\frac{dE}{dt} = \frac{1}{2} \int_D \left[u \tau_x^{xx} + u \tau_y^{xy} + v \tau_x^{xy} + v \tau_y^{yy} \right] dx dy \quad (7.1)$$

where D is the two-dimensional domain. Integrating by parts and neglecting surface terms (work done by external forces) yields:

$$\frac{dE}{dt} = -\frac{1}{2} \int_D \left[u_x \tau^{xx} + u_y \tau^{xy} + v_x \tau^{yx} + v_y \tau^{yy} \right] dx dy \quad (7.2)$$

Substituting the finite-scale subgrid stresses into (7.2) yields:

$$\begin{aligned} \frac{dE_{FS}}{dt} &= \frac{1}{6} \left(\frac{\Delta x}{2} \right)^2 \langle \bar{u}_x^3 \rangle + \frac{1}{6} \left(\frac{\Delta y}{2} \right)^2 \langle \bar{v}_y^3 \rangle \\ &+ \frac{1}{6} \left[\left(\frac{\Delta x}{2} \right)^2 - \left(\frac{\Delta y}{2} \right)^2 \right] \langle \bar{u}_x \bar{u}_y \bar{v}_x \rangle \end{aligned} \quad (7.3)$$

Here the brackets indicate spatial integration over the domain. Note that $\langle \bar{u}_x^3 \rangle < 0$ and $\langle \bar{v}_y^3 \rangle < 0$ by Kolmogorov's 4/5 law, while (in an isotropic flow) the other terms on the right-hand-side of (7.3) vanish. This shows that (in the absence of forces) kinetic energy is absolutely decreasing and the finite scale equations are globally stable.

Next, substituting the implicit subgrid stresses of the modified equation into (7.2) yields:

$$\frac{dE_{ME}}{dt} = \frac{1}{2} \left(\frac{\Delta x}{2} \right)^2 \left[\frac{1}{3} \langle \bar{u}_x^3 \rangle - \langle |\bar{u}_x^3| \rangle + \frac{1}{3} \langle \bar{u}_x \bar{v}_x^2 \rangle - \langle |\bar{u}_x| \bar{v}_x^2 \rangle \right]$$

$$\begin{aligned}
& + \frac{1}{2} \left(\frac{\Delta y}{2} \right)^2 \left[\frac{1}{3} \langle \bar{v}_y^3 \rangle - \langle |\bar{v}_y^3| \rangle + \frac{1}{3} \langle \bar{v}_y \bar{u}_y^2 \rangle - \langle |\bar{v}_y| \bar{u}_y^2 \rangle \right] \\
& - \frac{1}{3} \left(\frac{\Delta x}{2} \right)^2 [\langle \bar{v} \bar{v}_x \bar{u}_{xx} \rangle - \langle \bar{u} \bar{v}_x \bar{v}_{xx} \rangle] \\
& - \frac{1}{3} \left(\frac{\Delta y}{2} \right)^2 [\langle \bar{u} \bar{u}_y \bar{v}_{yy} \rangle - \langle \bar{v} \bar{u}_y \bar{u}_{yy} \rangle]
\end{aligned} \tag{7.4}$$

There are obvious similarities with (7.3), especially in terms of the scaling. However, there is also an important difference in the presence of terms like $(-\langle |\bar{u}_x^3| \rangle)$, which are absolutely negative (i.e., dissipative). The NFV methods in general, and MPDATA in particular, are nonlinearly stable by construction. This can be seen in (7.4) where several of the terms can be grouped to ensure that the integrands are negative definite – e.g., $\langle \frac{1}{3} \bar{u}_x^3 - |\bar{u}_x^3| \rangle < 0$. This is a different kind of stability from that of the finite-scale equations, as it does not depend on the solution. In the language of numerical analysis, the MPDATA equations are locally stable, whereas the finite-scale equations are globally stable.

Based on the energy analysis, the MPDATA implicit subgrid stress tensor can be written as the sum of two parts, one of which is absolutely dissipative and one of which closely corresponds to the self-similar terms of (5.6) and (5.7). In the case of MPDATA, the dissipative stress is similar to a tensor version of the common Smagorinsky model. As remarked in section 5, the nonlinear terms are the same as those of the self-similar Clark model [14]. Thus, *MPDATA has the form of a mixed LES model*.

We note that while the physics demands that the self-similar terms scale like Δx^2 and Δy^2 , there is no such requirement on the dissipative terms. Further, one might suppose that NFV schemes where the dissipative terms scale with higher powers than 2 might better represent backscatter (energy transfer from smaller to larger scales) and hence large scale variability. In fact, there are NFV schemes with higher-order dissipation; in particular, there exist versions of MPDATA with this property. However, at present, the hypothesis of improved results has not been validated.

8 Conclusion and Future Work

In this paper, we have discussed a new approach to numerically simulating high Reynolds' number flows. The strategy of Implicit Large Eddy Simulation (*ILES*) consists of applying

fluid solvers based on nonoscillatory finite volume (NFV) approximations directly to turbulent flows; the numerical algorithms provide effective modeling of the unresolved dynamics and no explicit subgrid scale turbulence models are needed or used.

The advantages of the approach are numerous.

- The same fluid solvers may be used for turbulent and for laminar flows. It is not necessary to know in advance whether a flow is turbulent, and transitions are handled seamlessly.
- NFV methods are a mainstream direction in the broader computational fluid dynamics community. There are many methods, offering a diversity of balances of accuracy and computational efficiency; however, all NFV methods are nonlinearly stable subject to time step restrictions.
- NFV methods are parameter free; they do not have to be modified from one application to another.

ILES has been shown to be effective in many diverse applications of numerical modeling. To our knowledge, there are no documented failures. Our main purpose in this paper has been to offer a rationale for this effectiveness, in the hopes of promoting its acceptance in the turbulence modeling community. This rationale will also serve to identify limits of the *ILES* approach and to guide further research. The main points of our argument are: (1) that the volume-averaged quantities that correspond to experimental measurements obey different equations than the continuum PDEs such as Navier-Stokes; (2) that these equations (termed finite-scale equations) can be derived and depend explicitly on the length scales that define the volume averages; and finally (3) that NFV methods accurately solve these finite-scale equations.

Some important issues remain to be resolved. On the practical side, there is as yet no definitive theory for constructing boundary conditions for the finite-scale equations. There is no comprehensive insight to the relative merits of different NFV schemes; such an understanding may lead to the design of new and more effective algorithms for turbulence modeling. More generally, the limits of applicability of the approach are not yet quantified. It is our hope that this article will help stimulate interest and further research in *ILES*

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